THE GEOMETRY OF TRIPLE LINKING

Herman Gluck - Abstract

Given a link L in R^3 with just two components $X = \{x(s): s \in S^1\}$ and $Y = \{y(t): t \in S^1\}$, their ***linking number*** can be defined as the degree of the ***Gauss map*** $f_L: S^1 \times S^1 \to S^2$, given by $f_L(s, t) = (y(t) - x(s)) / |y(t) - x(s)|$, or as the value of the ***Gauss integral***, $1/4\pi \int_{S^1 \times S^1} (dx/ds) \times (dy/dt) \bullet (x - y)/|x - y|^3 \, ds \, dt$.

The Gauss map is equivariant with respect to the action of an orientation-preserving isometry h of R^3, that is, $f_{h(L)} = h \circ f_L$, and the Gauss integrand is invariant with respect to this action, meaning that it is the same for L and for $h(L)$, attesting to the geometric naturality of both the map and the integral.

A consequence of this naturality is that the Gauss integral hides inside Ampere's Law of classical electrodynamics, and appears explicitly in the notions of ***helicity*** in plasma physics and fluid dynamics, and of the ***writhing number*** of a simple closed curve, used in molecular biology.

We quickly review the above ideas, constructions and applications, and then turn to the corresponding notions for three-component links L in Euclidean 3-space R^3 and the 3-sphere S^3.

In both cases we introduce generalizations $f_L: S^1 \times S^1 \times S^1 \to S^2$ of the Gauss map which share its essential features: simplicity, naturality in the above sense, and this time a transfer of the classical Milnor invariants for three-component links up to link homotopy to the classical Pontryagin invariants for the characteristic map f_L up to homotopy. Then, when the pairwise linking numbers are all zero, we use this map to derive an integral formula for Milnor's triple linking number which is a natural successor to the Gauss integral above. Core proofs are in terms of framed cobordism in three-manifolds.

Motivation comes from the Arnold-Khesin search for "higher order helicities" of plasma and fluid flows.

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